A NOTE ON SYMMETRIC MATRICES.*

By Gilbert Ames Bliss

In a recent paper Dickson† has given a very concise and elementary proof of the well-known fact that for a symmetric matrix of rank r at least one principal minor of order r is not zero. Wedderburn‡ deduced the same result for matrixes with real elements in proving, by a less elementary though very concise method, that the sum of the principal minors does not vanish under the same circumstances. The purpose of this note is to exhibit a theorem which includes both of the results just mentioned, and throws further light on the structure of a symmetric matrix.

Suppose that the order of a square matrix A is m, and let the combinations of r letters selected out of the set $(1, 2, \cdots, m)$ be denoted by the symbols P_i $(i = 1, 2, \cdots, n)$. The symbol A_{ik} will represent the minor of A of order r with rows corresponding to the numbers P_i and columns to the numbers P_k. The matrix (A_{ik}) is evidently symmetric if the original matrix A has this property.

For a symmetric matrix of rank r there exists a multiplier $M \neq 0$ and a set of numbers X_i $(i = 1, 2, \cdots, n)$ not all zero such that

$$A_{ij} = MX_iX_j \qquad (i, j = 1, 2, \cdots, n). \tag{1}$$

The justification of these statements depends upon the known fact, a proof of which is given below, that the ratios $A_{i1} : A_{i2} : \cdots : A_{in}$ $(i = 1, 2, \cdots, n)$ for a matrix of rank r or less are all the same.§ In other words there exists a set of numbers $X_1, X_2, \cdots, X_n$ not all zero, with a set of multipliers M_i $(i = 1, 2, \cdots, n)$ such that

$$A_{i1} = M_iX_1, \quad A_{i2} = M_iX_2, \quad \cdots, \quad A_{in} = M_iX_n \quad (i = 1, 2, \cdots, n).$$

Any row $A_{i1}, A_{i2}, \cdots, A_{in}$ with elements not all zero would be a set of numbers X with these properties. Suppose that X_k is different from zero. Then M is uniquely determined by the condition $A_{kk} = MX_k^2$, and it follows that the multiplier M_k of the kth row has the value MX_k, while the elements A_{ki} of that row have the values

$$A_{ki} = MX_kX_i = A_{ik} \qquad (i = 1, 2, \cdots, n).$$

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† *Annals of Mathematics*, vol. 15 (1913), page 27.

‡ Page 29 of the same journal.

§ Kowalewski, Einführung in die Determinantentheorie, page 122.

From the fact that this is also an expression for A_{ik} it follows at once that the multiplier M_i for the ith row has the value MX_i. The constant M can not be zero; otherwise the matrix would be of rank less than r.

If a matrix A, symmetric or not, has its ratios $A_{i1} : A_{i2} : \cdots : A_{in}$ identical, the same will be true after a row has been multiplied by a constant and added to another, and vice versa. The rank is also unchanged by this transformation. In proving that the ratios $A_{i1} : A_{i2} : \cdots : A_{in}$ are all equal, in the sense described above, for a matrix of rank r, it may be supposed without loss of generality that A_{11} is the principal minor of order r in the upper left-hand corner of A, and that it is different from zero. If the first r rows of A are multiplied by suitable constants and added to the remaining ones, it can be brought about that all the elements of A below A_{11} are zero. Since the resulting matrix has still rank r, it follows that all the other elements of the last $m - r$ rows must also be zero. But in that case the rows of elements $A_{i1}, A_{i2}, \cdots, A_{in}$ are all zero except when $i = 1$, and all the rows have the same ratio.

Since at least one X_k is different from zero it follows from the italicized theorem above that at least one principal minor is different from zero. If any minor A_{ik} of order r, which is not principal, is different from zero, then at least two principal minors do not vanish, namely, A_{ii} and A_{kk}. The sum of the principal minors of order r is evidently

$$\Sigma A_{ii} = M\Sigma X_i^2.$$

For the case of a matrix of real elements this is evidently different from zero, and all the principal minors which are different from zero have the same sign.

I am indebted to Mr. Meyer Gaba for the suggestion that the methods used above should also be applicable to Hermitian matrices for which any pair of elements symmetric with respect to the principal diagonal are conjugate imaginaries. This is in fact the case, and the minors of order r of a Hermitian matrix of rank r can be expressed in the form

$$A_{ik} = MX_i'X_k \qquad (i, k = 1, 2, \cdots, n),$$

where X_i' is the conjugate to X_i and M is real.

Any skew symmetric determinant of odd order is necessarily zero. Hence if a skew symmetric matrix is of odd rank r, all the principal minors of order r vanish. By an argument similar to that above it follows that for an element X_k different from zero the only value of M satisfying the relation $A_{kk} = MX_k^2$ is zero. Hence all the minors A_{ik} vanish. In case the rank r is even the minors have again the form (1).

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